An existence and uniqueness result for mean field games with congestion effect on graphs

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Abstract

This paper presents a general existence and uniqueness result for the mean field games equations on a graph ($\mathcal{G}$-MFG). In particular, our setting allows to take into account congestion effects as those initially evoked in [16] in a continuous framework or even non-local forms of congestion. These general congestion effects are particularly relevant in graphs in which the cost to move from on node to another may for instance depend on the proportion of players in both the source node and the target node. Existence is proved using a priori estimates. Uniqueness is obtained through the usual algebraic manipulations initially proposed in [14], or [16] in the case of congestion, and we propose a new criterion to ensure uniqueness that allows for hamiltonian functions with a more complex structure.

Introduction

Mean field games have been introduced in 2006 by J.-M. Lasry and P.-L. Lions [13, 14, 15] as the limit of some stochastic differential games when the number of players increased toward infinity.

Since then, many applications of mean field games have been proposed, particularly in economics (see for instance [7], [11], or [12]) and an important effort has been made to solve the partial differential equations associated to mean field games when both time and the state space are continuous (see [1], [2], [10], [12], etc).

In this paper, we consider, as in [9], a mean field game on a graph and we prove the existence and uniqueness of a solution to the discrete counterpart of the usual MFG equations. Our setting is more general than the one developed in [9] and allows for non-local congestion effects. In other words, hamiltonian functions are very general and depend on the whole distribution of the players' position on the graph. Our existence result is obtained by a Schauder fixed point theorem and the uniqueness result takes the form of an algebraic condition to ensure the positiveness of a matrix.

In the first section, we introduce the $\mathcal{G}$-MFG equations in the case of a graph with congestion effects. In section 2, we prove the existence of a C^1 solution to the $\mathcal{G}$-MFG equations using a priori bounds obtained through a comparison principle. In section 3, we provide a condition to ensure uniqueness of this solution.

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1 Mean field games

We consider a directed graph $\mathcal{G}$ whose nodes are indexed by integers from 1 to N. For each node $i \in \mathcal{N} = \{1, \dots, N\}$ we introduce $\mathcal{V}(i) \subset \mathcal{N} \setminus \{i\}$ the set of nodes j for which a directed edge exists from i to j. The cardinal of this set is denoted d_i and called the out-degree of the node i. Reciprocally, we denote $\mathcal{V}^{-1}(i) \subset \mathcal{N} \setminus \{i\}$ the set of nodes j for which a directed edge exists from j to i.

This graph $\mathcal{G}$ is going to be the state space of our mean field game and we suppose that there is a continuum of anonymous and identical players of size 1[1].

Instantaneous transition probabilities at time t are described by a collection of functions $\lambda_t(i, \cdot) : \mathcal{V}(i) \to \mathbb{R}_+$ (for each node $i \in \mathcal{N}$).

Then, the distribution of the players' position on the graph $\mathcal{G}$ is given by a function $t \mapsto m(t) = (m(t, 1), \dots, m(t, N))$.

Each player is able to decide on the values of the transition probabilities at time t, up to cost, and we assume that the instantaneous payoff of a player is given by $-\mathcal{L}(i, (\lambda_{i,j})_{j \in \mathcal{V}(i)}, m)$ if he is in position i, if he sets the value of $\lambda(i, j)$ to $\lambda_{i,j}$ (for all $j \in \mathcal{V}(i)$), and if the distribution of the players' position is m.

This framework generalizes what was done in [9] in which $\mathcal{L}$ was of the form:

$$\mathcal{L}(i, (\lambda_{i,j})_{j \in \mathcal{V}(i)}, m) = L(i, (\lambda_{i,j})_{j \in \mathcal{V}(i)}) - f(i, m)$$

Here, we allow for more general forms and we have in mind applications to congestion. In such cases, the price to pay to move from one particular node to another depends on the distribution of the players in a non-additive way.

The assumptions made on the functions $\mathcal{L}(i, \cdot, \cdot)$ are the following:

- Continuity: $\forall i \in \mathcal{N}, \mathcal{L}(i, \cdot, \cdot)$ is a continuous function from $\mathbb{R}_+^{d_i} \times \Omega$ to $\mathbb{R}$ where $\Omega \subset \mathbb{R}^N$ is a domain containing $\mathcal{P}_N = \{(x_1, \dots, x_N) \in [0, 1]^n, \sum_{i=1}^{N} x_i = 1\}$.

- Asymptotic super-linearity: $\forall i \in \mathcal{N}, \forall m \in \Omega, \quad \lim_{\lambda \in \mathbb{R}_+^{d_i}, |\lambda| \to +\infty} \frac{\mathcal{L}(i, \lambda, m)}{|\lambda|} = +\infty$.

Then, we can define the hamiltonian functions:

$$\forall i \in \mathcal{N}, p \in \mathbb{R}^{d_i}, m \in \Omega \mapsto \mathcal{H}(i, p, m) = \sup_{\lambda \in \mathbb{R}_+^{d_i}} \lambda \cdot p - \mathcal{L}(i, \lambda, m)$$

and we assume that:

- $\forall i \in \mathcal{N}, \mathcal{H}(i, \cdot, \cdot)$ is a continuous function.

- $\forall i \in \mathcal{N}, \forall m \in \Omega, \mathcal{H}(i, \cdot, m)$ is a C^1 function with:

$$\frac{\partial \mathcal{H}}{\partial p}(i, p, m) = \text{argmax}_{\lambda \in \mathbb{R}_+^{d_i}} \lambda \cdot p - \mathcal{L}(i, \lambda, m)$$

  Remark: This assumption is satisfied as soon as $\forall i \in \mathcal{N}, \forall m \in \Omega, \mathcal{L}(i, \cdot, m)$ is a C^2 strongly convex function or if $\mathcal{L}(i, \lambda, m) = \sum_{j \in \mathcal{V}(i)} C(i, j, \lambda_{i,j}, m)$ with each of the functions $C(i, j, \cdot, m)$ a strictly convex C^2 function.

- $\forall i \in \mathcal{N}, \forall j \in \mathcal{V}(i), \frac{\partial \mathcal{H}}{\partial p_j}(i, \cdot, \cdot)$ is a continuous function.

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[1]To see our setting as the limit of a finite state space M-player-games as $M \to +\infty$, one can refer to a paper by Diogo Gomes et al. ([6]), justifying our direct passage to a continuum of players.

Now, let us define the $\mathcal{G}$-MFG equations associated to the above mean field game with a terminal payoff given by functions $g(i, \cdot)$ continuous on Ω and an initial distribution $m^0 \in \mathcal{P}_N$:

Definition 1 (The $\mathcal{G}$-MFG equations). *The $\mathcal{G}$-MFG equations consist in a system of $2N$ equations, the unknown being $t \in [0, T] \mapsto (u(t, 1), \ldots, u(t, N), m(t, 1), \ldots, m(t, N))$.*

The first half of these equations are Hamilton-Jacobi equations and consist in the following system:

$$\forall i \in \mathcal{N}, \quad \frac{d}{dt}u(t, i) + \mathcal{H}\left(i, (u(t, j) - u(t, i))_{j \in \mathcal{V}(i)}, m(t, 1), \ldots, m(t, N)\right) = 0$$

with $u(T, i) = g(i, m(t, 1), \ldots, m(t, N))$.

The second half of these equations are forward transport equations:

$$\forall i \in \mathcal{N}, \quad \frac{d}{dt}m(t, i) = \sum_{j \in \mathcal{V}^{-1}(i)} \frac{\partial \mathcal{H}(j, \cdot, m(t, 1), \ldots, m(t, N))}{\partial p_i} \left((u(t, k) - u(t, j))_{k \in \mathcal{V}(j)}\right) m(t, j)$$

$$- \sum_{j \in \mathcal{V}(i)} \frac{\partial \mathcal{H}(i, \cdot, m(t, 1), \ldots, m(t, N))}{\partial p_j} \left((u(t, k) - u(t, i))_{k \in \mathcal{V}(i)}\right) m(t, i)$$

with $(m(0, 1), \ldots, m(0, N)) = m^0$.

In what follows, existence and uniqueness of solutions to the $\mathcal{G}$-MFG equations are going to be tackled. We first start with existence and our proof is based on a Schauder fixed-point argument and a priori estimates to obtain compactness. Then we will present a criterion to ensure uniqueness of C^1 solutions.

2 Existence of a solution

For the existence result, we first start with a Lemma stating that, for a fixed m, the N Hamilton-Jacobi equations amongst the $\mathcal{G}$-MFG equations obey a comparison principle:

Lemma 1 (Comparison principle). *Let $m : [0, T] \rightarrow \mathcal{P}_N$ be a continuous function. Let $u : t \in [0, T] \mapsto (u(t, 1), \ldots, u(t, N))$ be a C^1 function that verifies:*

$$\forall i \in \mathcal{N}, \quad -\frac{d}{dt}u(t, i) - \mathcal{H}\left(i, (u(t, j) - u(t, i))_{j \in \mathcal{V}(i)}, m(t, 1), \ldots, m(t, N)\right) \leq 0$$

with $u(T, i) \leq g(i, m(T, 1), \ldots, m(T, N))$.

Let $v : t \in [0, T] \mapsto (v(t, 1), \ldots, v(t, N))$ be a C^1 function that verifies:

$$\forall i \in \mathcal{N}, \quad -\frac{d}{dt}v(t, i) - \mathcal{H}\left(i, (v(t, j) - v(t, i))_{j \in \mathcal{V}(i)}, m(t, 1), \ldots, m(t, N)\right) \geq 0$$

with $v(T, i) \geq g(i, m(T, 1), \ldots, m(T, N))$.

Then, $\forall i \in \mathcal{N}, \forall t \in [0, T], v(t, i) \geq u(t, i)$.

Proof:

Let us consider for a given $\epsilon > 0$, a point $(t^*, i^*) \in [0,T] \times \mathcal{N}$ such that

$$u(t^*, i^*) - v(t^*, i^*) - \epsilon(T - t^*) = \max_{(t,i) \in [0,T] \times \mathcal{N}} u(t,i) - v(t,i) - \epsilon(T - t)$$

If $t^* \in [0,T)$, then $\frac{d}{dt}\left(u(t,i^*) - v(t,i^*) - \epsilon(T - t)\right)\big|_{t=t^*} \leq 0$. Also, by definition of (t^*, i^*), $\forall j \in \mathcal{V}(i^*)$, $u(t^*, i^*) - v(t^*, i^*) \geq u(t^*, j) - v(t^*, j)$ and hence, by definition of $\mathcal{H}(i^*, \cdot, \cdot)$:

$$\mathcal{H}\left(i^*, (v(t^*, j) - v(t^*, i^*))_{j \in \mathcal{V}(i^*)}, m(t^*, 1), \ldots, m(t^*, N)\right)$$

$$\geq \mathcal{H}\left(i^*, (u(t^*, j) - u(t^*, i^*))_{j \in \mathcal{V}(i^*)}, m(t^*, 1), \ldots, m(t^*, N)\right)$$

Combining these inequalities we get:

$$-\frac{d}{dt}u(t^*, i^*) - \mathcal{H}\left(i^*, (u(t^*, j) - u(t^*, i^*))_{j \in \mathcal{V}(i^*)}, m(t^*, 1), \ldots, m(t^*, N)\right)$$

$$\geq -\frac{d}{dt}v(t^*, i^*) - \mathcal{H}\left(i^*, (v(t^*, j) - v(t^*, i^*))_{j \in \mathcal{V}(i^*)}, m(t^*, 1), \ldots, m(t^*, N)\right) + \epsilon$$

But this is in contradiction with the hypotheses on u and v.

Hence $t^* = T$ and $\max_{(t,i) \in [0,T] \times \mathcal{N}} u(t,i) - v(t,i) - \epsilon(T - t) \leq 0$ because of the assumptions on $u(T, i)$ and $v(T, i)$.

This being true for any $\epsilon > 0$, we have that $\max_{(t,i) \in [0,T] \times \mathcal{N}} u(t,i) - v(t,i) \leq 0$. $\qquad\square$

This lemma allows to provide a bound to any solution u of the N Hamilton-Jacobi equations and this bound is then used to obtain compactness in order to apply Schauder's fixed point theorem.

Proposition 1 (Existence). *Under the assumptions made in section 1, there exists a C^1 solution (u, m) of the $\mathcal{G}$-MFG equations.*

Proof:

Let $m : [0,T] \to \mathcal{P}_N$ be a continuous function.

Let then consider the solution $u : t \in [0,T] \mapsto (u(t,1), \ldots, u(t,N))$ to the Hamilton-Jacobi equations:

$$\forall i \in \mathcal{N}, \quad \frac{d}{dt}u(t,i) + \mathcal{H}\left(i, (u(t,j) - u(t,i))_{j \in \mathcal{V}(i)}, m(t,1), \ldots, m(t,N)\right) = 0$$

with $u(T, i) = g(i, m(t,1), \ldots, m(t,N))$.

This function u is a well defined C^1 function with the following bound coming from the above lemma:

$$\sup_{i \in \mathcal{N}} \|u(\cdot, i)\|_\infty \leq \sup_{i \in \mathcal{N}} \|g(i, \cdot)\|_\infty + T \sup_{i \in \mathcal{N}, m \in \mathcal{P}_N} |\mathcal{H}(i, 0, m)|$$

Using this bound and the assumptions on $\mathcal{H}$ we can define a function $\tilde{m} : [0,T] \to \mathcal{P}_N$ by:

$$\forall i \in \mathcal{N}, \quad \frac{d}{dt}\tilde{m}(t,i) = \sum_{j \in \mathcal{V}^{-1}(i)} \frac{\partial \mathcal{H}(j, \cdot, m(t,1), \ldots, m(t,N))}{\partial p_i}\left((u(t,k) - u(t,j))_{k \in \mathcal{V}(j)}\right) \tilde{m}(t,j)$$

$$- \sum_{j \in \mathcal{V}(i)} \frac{\partial \mathcal{H}(i, \cdot, m(t,1), \ldots, m(t,N))}{\partial p_j}\left((u(t,k) - u(t,i))_{k \in \mathcal{V}(i)}\right) \tilde{m}(t,i)$$

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with $(\tilde{m}(0,1), \ldots, \tilde{m}(0,N)) = m^0 \in \mathcal{P}_N$.

$\frac{d\tilde{m}}{dt}$ is bounded, the bounds depending only on the functions $g(i,\cdot)$ and $\mathcal{H}(i,\cdot,\cdot)$, $i \in \mathcal{N}$.

As a consequence, if we define $\Theta : m \in C([0,T], \mathcal{P}_N) \mapsto \tilde{m} \in C([0,T], \mathcal{P}_N)$, Θ is a continuous function (from classical ODEs theory) with $\Theta(C([0,T], \mathcal{P}_N))$ a relatively compact set (because of Ascoli's Theorem and the uniform Lipschitz property we just obtained).

Hence, because $C([0,T], \mathcal{P}_N)$ is convex, by Schauder's fixed point theorem, there exists a fixed point m to Θ. If we then consider u associated to m by the Hamilton-Jacobi equations as above, (u,m) is a C^1 solution to the $\mathcal{G}$-MFG equations. $\qquad \square$

3 Uniqueness

Coming now to uniqueness, we use the same algebraic manipulations as in [14, 15] or [9] in the case of graphs and we obtain a criterion close to the one obtained in [16] but adapted to graphs and generalized to non-local congestion.

Proposition 2 (Uniqueness). *Assume that g is such that:*

$$\forall (m,\mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^{N} (g(i,m_1,\ldots,m_N) - g(i,\mu_1,\ldots,\mu_N))(m_i - \mu_i) \geq 0 \implies m = \mu$$

Assume that the hamiltonian functions can be written as:

$$\forall i \in \mathcal{N}, \forall p \in \mathbb{R}^{d_i}, \forall m \in \Omega, \mathcal{H}(i,p,m) = \mathcal{H}_c(i,p,m) + f(i,m)$$

with $\forall i \in \mathcal{N}$, $f(i,\cdot)$ a continuous function satisfying

$$\forall (m,\mu) \in \mathcal{P}_N \times \mathcal{P}_N, \sum_{i=1}^{N} (f(i,m_1,\ldots,m_N) - f(i,\mu_1,\ldots,\mu_N))(m_i - \mu_i) \geq 0 \implies m = \mu$$

and $\forall i \in \mathcal{N}$, $\mathcal{H}_c(i,\cdot,\cdot)$ a C^1 function with $\forall j \in \mathcal{V}(i)$, $\frac{\partial \mathcal{H}_c}{\partial p_j}(i,\cdot,\cdot)$ a C^1 function on $\mathbb{R}^n \times \Omega$

Now, let us define $A : (q_1,\ldots,q_N,m) \in \prod_{i=1}^{N} \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto (\alpha_{ij}(q_i,m))_{i,j} \in \mathcal{M}_N$ defined by:

$$\alpha_{ij}(q_i,m) = -\frac{\partial \mathcal{H}_c}{\partial m_j}(i,q_i,m)$$

Let us also define, $\forall i \in \mathcal{N}$, $B^i : (q_i,m) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left(\beta_{jk}^i(q_i,m)\right)_{j,k} \in \mathcal{M}_{N,d_i}$ defined by:

$$\beta_{jk}^i(q_i,m) = m_i \frac{\partial^2 \mathcal{H}_c}{\partial m_j \partial q_{ik}}(i,q_i,m)$$

Let us also define, $\forall i \in \mathcal{N}$, $C^i : (q_i,m) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left(\gamma_{jk}^i(q_i,m)\right)_{j,k} \in \mathcal{M}_{d_i,N}$ defined by:

$$\gamma_{jk}^i(q_i,m) = m_i \frac{\partial^2 \mathcal{H}_c}{\partial m_k \partial q_{ij}}(i,q_i,m)$$

Let us finally define, $\forall i \in \mathcal{N}$, $D^i : (q_i,m) \in \mathbb{R}^{d_i} \times \mathcal{P}_N \mapsto \left(\delta_{jk}^i(q_i,m)\right)_{j,k} \in \mathcal{M}_{d_i}$ defined by:

$$\delta_{jk}^i(q_i,m) = m_i \frac{\partial^2 \mathcal{H}_c}{\partial q_{ij} \partial q_{ik}}(i,q_i,m)$$

Assume that $\forall (q_1, \ldots, q_N, m) \in \prod_{i=1}^{N} \mathbb{R}^{d_i} \times \mathcal{P}_N$:

$$M(q_1, \ldots, q_N, m) = \begin{pmatrix} A(q_1, \ldots, q_N, m) & B^1(q_1, m) & \cdots & \cdots & \cdots & B^N(q_N, m) \\ C^1(q_1, m) & D^1(q_1, m) & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ C^N(q_N, m) & 0 & \cdots & \cdots & 0 & D^N(q_N, m) \end{pmatrix} \geq 0$$

Then, if $(\widehat{u}, \widehat{m})$ *and* $(\tilde{u}, \tilde{m})$ *are two* C^1 *solutions of the* $\mathcal{G}$-*MFG equations, we have* $\widehat{m} = \tilde{m}$ *and* $\widehat{u} = \tilde{u}$.

Proof:

The proof of this result consists in computing in two different ways the value of

$$I = \int_0^T \sum_{i=1}^N \frac{d}{dt} \left((\widehat{u}(t,i) - \tilde{u}(t,i))(\widehat{m}(t,i) - \tilde{m}(t,i)) \right) dt$$

We first know directly that

$$I = \sum_{i=1}^N (g(i, \widehat{m}(T)) - g(i, \tilde{m}(T)))(\widehat{m}(T,i) - \tilde{m}(T,i))$$

Now, differentiating the product we get, that:

$$I = -\int_0^T \sum_{i=1}^N (f(i, \widehat{m}(t)) - f(i, \tilde{m}(t)))(\widehat{m}(t,i) - \tilde{m}(t,i)) dt$$

$$+ \int_0^T \sum_{i=1}^N (\widehat{m}(t,i) - \tilde{m}(t,i)) \left[\mathcal{H}_c(i, (\tilde{u}(t,k) - \tilde{u}(t,i))_{k \in \mathcal{V}(i)}, \tilde{m}(t)) - \mathcal{H}_c(i, (\widehat{u}(t,k) - \widehat{u}(t,i))_{k \in \mathcal{V}(i)}, \widehat{m}(t)) \right] dt$$

$$+ \int_0^T \sum_{i=1}^N (\widehat{u}(t,i) - \tilde{u}(t,i)) \left[\sum_{j \in \mathcal{V}^{-1}(i)} \frac{\mathcal{H}_c}{\partial p_i}(j, (\widehat{u}(t,k) - \widehat{u}(t,j))_{k \in \mathcal{V}(j)}, \widehat{m}(t)) \widehat{m}(t,j) \right.$$

$$- \sum_{j \in \mathcal{V}(i)} \frac{\mathcal{H}_c}{\partial p_j}(i, (\widehat{u}(t,k) - \widehat{u}(t,i))_{k \in \mathcal{V}(i)}, \widehat{m}(t)) \widehat{m}(t,i) - \sum_{j \in \mathcal{V}^{-1}(i)} \frac{\mathcal{H}_c}{\partial p_i}(j, (\tilde{u}(t,k) - \tilde{u}(t,j))_{k \in \mathcal{V}(j)}, \tilde{m}(t)) \tilde{m}(t,j)$$

$$\left. + \sum_{j \in \mathcal{V}(i)} \frac{\mathcal{H}_c}{\partial p_j}(i, (\tilde{u}(t,k) - \tilde{u}(t,i))_{k \in \mathcal{V}(i)}, \tilde{m}(t)) \tilde{m}(t,i) \right] dt$$

After reordering the terms we get:

$$I = -\int_0^T \sum_{i=1}^N (f(i, \widehat{m}(t)) - f(i, \tilde{m}(t)))(\widehat{m}(t,i) - \tilde{m}(t,i)) dt$$

$$+ \int_0^T \sum_{i=1}^N (\widehat{m}(t,i) - \tilde{m}(t,i)) \left[\mathcal{H}_c(i, (\tilde{u}(t,k) - \tilde{u}(t,i))_{k \in \mathcal{V}(i)}, \tilde{m}(t)) - \mathcal{H}_c(i, (\widehat{u}(t,k) - \widehat{u}(t,i))_{k \in \mathcal{V}(i)}, \widehat{m}(t)) \right] dt$$

$$+ \int_0^T \sum_{i=1}^N \widehat{m}(t,i) \sum_{j \in \mathcal{V}(i)} ((\widehat{u}(t,j) - \tilde{u}(t,j)) - (\widehat{u}(t,i) - \tilde{u}(t,i))) \frac{\partial \mathcal{H}_c}{\partial p_j} \left(i, (\widehat{u}(t,k) - \widehat{u}(t,i))_{k \in \mathcal{V}(i)}, \widehat{m}(t) \right) dt$$

$$-\int_0^T \sum_{i=1}^N \tilde{m}(t,i) \sum_{j \in \mathcal{V}(i)} ((\widehat{u}(t,j)-\tilde{u}(t,j))-(\widehat{u}(t,i)-\tilde{u}(t,i))) \frac{\partial \mathcal{H}_c}{\partial p_j} \left(i,(\tilde{u}(t,k)-\tilde{u}(t,i))_{k \in \mathcal{V}(i)}, \tilde{m}(t)\right) dt$$

i.e.:

$$I = -\int_0^T \sum_{i=1}^N (f(i,\widehat{m}(t)) - f(i,\tilde{m}(t)))(\widehat{m}(t,i) - \tilde{m}(t,i))dt + J$$

where

$$J = \int_0^T \sum_{i=1}^N (\widehat{m}(t,i) - \tilde{m}(t,i)) \left[\mathcal{H}_c(i,(\tilde{u}(t,k)-\tilde{u}(t,i))_{k \in \mathcal{V}(i)}, \tilde{m}(t)) - \mathcal{H}_c(i,(\widehat{u}(t,k)-\widehat{u}(t,i))_{k \in \mathcal{V}(i)}, \widehat{m}(t)) \right] dt$$

$$+ \int_0^T \sum_{i=1}^N \sum_{j \in \mathcal{V}(i)} ((\widehat{u}(t,j)-\tilde{u}(t,j))-(\widehat{u}(t,i)-\tilde{u}(t,i))) \left[\widehat{m}(t,i) \frac{\partial \mathcal{H}_c}{\partial p_j} \left(i,(\widehat{u}(t,k) - \widehat{u}(t,i))_{k \in \mathcal{V}(i)}, \widehat{m}(t)\right) \right.$$

$$\left. -\tilde{m}(t,i) \frac{\partial \mathcal{H}_c}{\partial p_j} \left(i,(\tilde{u}(t,k) - \tilde{u}(t,i))_{k \in \mathcal{V}(i)}, \tilde{m}(t)\right) \right] dt$$

Now, if we denote $u^\theta(t) = \tilde{u}(t) + \theta(\widehat{u}(t) - \tilde{u}(t))$ and $m^\theta(t) = \tilde{m}(t) + \theta(\widehat{m}(t) - \tilde{m}(t))$, then we have:

$$J = \int_0^T \sum_{i=1}^N (\widehat{m}(t,i) - \tilde{m}(t,i)) \int_0^1 \left[\sum_{j \in \mathcal{V}(i)} -\frac{\partial \mathcal{H}_c}{\partial p_j}(i,(u^\theta(t,k) - u^\theta(t,i))_{k \in \mathcal{V}(i)}, m^\theta(t)) \right.$$

$$\times ((\widehat{u}(t,j)-\tilde{u}(t,j))-(\widehat{u}(t,i)-\tilde{u}(t,i))) + \sum_{j=1}^N -\frac{\partial \mathcal{H}_c}{\partial m_j}(i,(u^\theta(t,k)-u^\theta(t,i))_{k \in \mathcal{V}(i)}, m^\theta(t))(\widehat{m}(t,j)-\tilde{m}(t,j)) \right] d\theta dt$$

$$+ \int_0^T \sum_{i=1}^N \sum_{j \in \mathcal{V}(i)} \int_0^1 \left[(\widehat{m}(t,i) - \tilde{m}(t,i)) \frac{\partial \mathcal{H}_c}{\partial p_j} \left(i,(u^\theta(t,k) - u^\theta(t,i))_{k \in \mathcal{V}(i)}, m^\theta(t)\right) \right.$$

$$+ m^\theta(t,i) \sum_{l \in \mathcal{V}(i)} ((\widehat{u}(t,l) - \tilde{u}(t,l)) - (\widehat{u}(t,i) - \tilde{u}(t,i))) \frac{\partial^2 \mathcal{H}_c}{\partial p_l \partial p_j} \left(i,(\tilde{u}(t,k)-\tilde{u}(t,i))_{k \in \mathcal{V}(i)}, \tilde{m}(t)\right)$$

$$+ m^\theta(t,i) \sum_{l=1}^N (\widehat{m}(t,l) - \tilde{m}(t,l)) \frac{\partial^2 \mathcal{H}_c}{\partial m_l \partial p_j} \left(i,(\tilde{u}(t,k) - \tilde{u}(t,i))_{k \in \mathcal{V}(i)}, \tilde{m}(t)\right) \right] d\theta$$

$$\times ((\widehat{u}(t,j) - \tilde{u}(t,j)) - (\widehat{u}(t,i) - \tilde{u}(t,i)))dt$$

Now, we see that the first group of terms cancels with the third one and we can write J as:

$$J = \int_0^T \int_0^1 V(t) M((u^\theta(t,k)-u^\theta(t,1))_{k \in \mathcal{V}(1)}, \ldots, (u^\theta(t,k)-u^\theta(t,N))_{k \in \mathcal{V}(N)}, m^\theta(t)) V(t)' d\theta dt$$

where

$$V(t) = (\widehat{m}(t) - \tilde{m}(t), ((\widehat{u}(t,k) - \tilde{u}(t,k)) - (\widehat{u}(t,1) - \tilde{u}(t,1)))_{k \in \mathcal{V}(1)}, \ldots,$$

$$((\widehat{u}(t,k) - \tilde{u}(t,k)) - (\widehat{u}(t,N) - \tilde{u}(t,N)))_{k \in \mathcal{V}(N)})$$

Hence $J \geq 0$ and we have:

$$\int_0^T \sum_{i=1}^N (f(i,\widehat{m}(t)) - f(i,\tilde{m}(t)))(\widehat{m}(t,i) - \tilde{m}(t,i))dt$$

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$$+ \sum_{i=1}^{N} (g(i, \widehat{m}(T)) - g(i, \tilde{m}(T)))(\widehat{m}(T, i) - \tilde{m}(T, i)) \geq 0$$

Using the hypotheses on f and g we get $\widehat{m} = \tilde{m}$. The comparison principle stated in Lemma 1 then brings $\widehat{u} = \tilde{u}$ and the result is proved. □

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